

Simple dispersion law for arbitrary sequences of dispersive optics

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We give a simple general formula for the total angular dispersion due to multiple arbitrary dispersive elements in a series. It is simply the sum of the individual elements' angular dispersions but with each divided by the total spatial magnification afterward (or, equivalently, multiplied by the total angular magnification afterward). © 2010 Optical Society of America

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1. Sequences of Dispersive Elements

Multiple dispersive elements in sequence are commonly used for beam magnification and wavelength tuning in lasers [1–6], ultrashort-laser-pulse compression [7–14], and applications in which a single element yields too little or too much angular dispersion or magnification. Also, many devices involve the use of multiple passes through a single dispersive element or combinations of elements [11,14], which is effectively the same problem.

Unfortunately, formulas in the literature for modeling these devices are quite complex and unintuitive, and they cannot be generalized to cover all the dispersive elements; see, for example, [15].

Here we present a very simple equation for the total angular dispersion introduced by a sequence of arbitrary dispersive elements, such as prisms, gratings, and etalons, in terms of their respective angular dispersions and spatial or angular magnifications, which seems to have escaped notice in the literature and textbooks [16–23]. It is a generalization of a result first obtained by Trebino for a sequence of prisms [4]. We derive this more general result using the Kostenbauder matrix formalism. In addition, we use it to model a prism-grating pulse compressor, in

which a prism and grating are quadruple passed, an otherwise prohibitively difficult problem.

2. Theory

We first note that the expression for the dispersion [18], D , of a single general prism is

$$D = \left[\frac{dn \sin(\phi)}{d\lambda \cos(\theta)} / \frac{\cos(\gamma)}{\cos(\beta)} \right] + \frac{dn \sin(\beta)}{d\lambda \cos(\phi)}, \quad (1)$$

where n is the refractive index, λ is the wavelength, θ is the incidence angle of the beam at the entrance face, ϕ is the transmitted angle of the beam at the entrance face, β is the incidence angle of the beam at the exit face, and γ is the transmitted angle of the beam at the exit face; see Fig. 1. Note that $\cos(\gamma)/\cos(\beta)$ is the beam spatial magnification at the exit face, and the other quantities are the dispersions at the two individual faces. Thus, this expression can be rewritten very simply as the sum of the dispersions at the first (d_1) and second (d_2) faces, but with that of the first face divided by the spatial magnification that occurs at the second (m_2):

$$D = \frac{d_1}{m_2} + d_2. \quad (2)$$

In fact, the same is true for the dispersions and magnifications of sequences of whole prisms, and Trebino

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has shown (by simplifying an expression in [5] written in terms of the angles) that the total dispersion due to a sequence of N prisms is given by [4]

$$D_{\text{total}} = \frac{D_1}{M_2 M_3 \cdots M_N} + \frac{D_2}{M_3 M_4 \cdots M_N} + \cdots + D_N, \quad (3)$$

where D_i and M_i are the angular dispersion and spatial magnification of the i th prism, respectively. In this expression and in what follows, the dispersion is taken to be negative if the element is inverted.

We can easily generalize this expression to arbitrary numbers of arbitrary dispersive elements. To do so, recall that the propagation of an ultrashort pulse through a sequence of dispersive devices can be modeled using the Kostenbauder matrix (K matrix) formalism [24]. In this formalism, each dispersive element is represented by a 4×4 ray-pulse matrix, analogous to the well-known 2×2 ray-matrix formalism for geometric optics but now including time and frequency, in addition to position and angle. K matrices also allow for spatiotemporal distortions, such as angular dispersion. An entire optical system consisting of multiple geometric and dispersive elements can be described by the product of the individual 4×4 K matrices.

In general, a K matrix for a linear, time-invariant, and nonfocusing dispersive device will have certain elements that are always zero and one, as shown by Kostenbauder. The other matrix elements depend on the device parameters. These matrix elements correspond to the magnification (M), position versus angle (B), dispersion (D), spatial chirp (E), group-delay dispersion (I), time versus angle (H), and the pulse-front tilt (DM/λ_0) introduced by the optic:

$$K = \begin{bmatrix} M & B & 0 & E \\ 0 & 1/M & 0 & D \\ DM/\lambda_0 & H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4)$$

Consider first a grating as the dispersive element. The beam magnification by the grating is given in

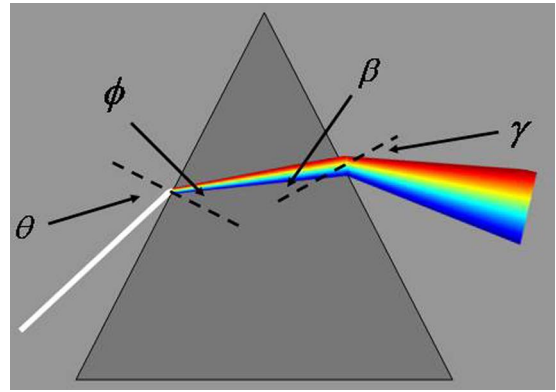


Fig. 1. (Color online) Prism, beam, and their relevant angles; dashed lines are normals to the prism faces.

terms of the incidence angle, θ , and the diffraction angle, γ , both measured from the surface normal:

$$M = -\frac{\cos \gamma}{\cos \theta}. \quad (5)$$

The angular dispersion introduced by a grating is given by

$$D = \frac{\sin \gamma - \sin \theta}{f_0 \cos \gamma}, \quad (6)$$

where f_0 is the center frequency of the pulse in cycles per second.

In terms of these quantities, the K matrix for a grating is

$$K_{\text{grating}} = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & 1/M & 0 & D \\ DM/\lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

Now consider a pulse diffracted by N gratings sequentially. In the following, we omit the K matrices representing free-space propagation because they add no angular dispersion or magnification and, therefore, contribute nothing to the quantities of interest. The K matrix representing this system is obtained by multiplying together the matrices of each individual grating:

$$\begin{aligned} & \begin{bmatrix} M_N & 0 & 0 & 0 \\ 0 & 1/M_N & 0 & D_N \\ D_N M_N / \lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & 1/M_1 & 0 & D_1 \\ D_1 M_1 / \lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & = \begin{bmatrix} \prod_{n=1}^N M_n & 0 & 0 & 0 \\ 0 & 1 / \prod_{n=1}^N M_n & 0 & \sum_{n=1}^N D_n / \prod_{p=n+1}^N M_p \\ \sum_{n=1}^N D_n \prod_{p=1}^n M_p / \lambda_0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (8) \end{aligned}$$

This expression can easily be proved using mathematical induction.

From the above K matrix for the system of gratings, the total dispersion is given as

$$D_{\text{total}} = \sum_{n=1}^N \frac{D_n}{\prod_{p=n+1}^N M_p}, \quad (9)$$

which is the dispersion expression for multiple prisms first derived by Trebin [4].

Next, we generalize this result to an arbitrary sequence of potentially different dispersive devices (see Fig. 2), including arbitrary prisms, reflection or transmission gratings, tilted interfaces, or etalons.

As before, the K matrix for the entire system of dispersive elements is given by the product of the K matrices of each individual element. In this product, we again ignore the free-space propagation between the elements because it does not contribute any dispersion or magnification. This calculation is shown below, and as before it can be proved by mathematical induction:

$$\begin{aligned} & \begin{bmatrix} M_N & B_N & 0 & E_N \\ 0 & 1/M_N & 0 & D_N \\ D_N M_N / \lambda_0 & H_N & 1 & I_N \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} M_1 & B_1 & 0 & E_1 \\ 0 & 1/M_1 & 0 & D_1 \\ D_1 M_1 / \lambda_0 & H_1 & 1 & I_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & = \begin{bmatrix} \prod_{n=1}^N M_n & B_{\text{total}} & 0 & E_{\text{total}} \\ 0 & 1 / \prod_{n=1}^N M_n & 0 & \sum_{n=1}^N D_n / \prod_{p=n+1}^N M_p \\ \sum_{n=1}^N D_n \prod_{p=1}^n M_p / \lambda_0 & H_{\text{total}} & 1 & I_{\text{total}} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (10)$$

Examining the elements of the above matrix, we see that the total magnification and angular dispersion of the system are given by

$$M_{\text{total}} = M_1 M_2 M_3 \cdots M_N, \quad (11)$$

$$D_{\text{total}} = \frac{D_1}{M_2 M_3 \cdots M_N} + \frac{D_2}{M_3 M_4 \cdots M_N} + \cdots + D_N. \quad (12)$$

Finally, we note that the beam angular magnification, $\mu_i = 1/M_i$, is the reciprocal of the spatial magnification; hence, this expression can be written perhaps even more simply and intuitively in terms of this latter quantity as

$$D_{\text{total}} = D_1 \mu_2 \mu_3 \cdots \mu_N + D_2 \mu_3 \mu_4 \cdots \mu_N + \cdots + D_N. \quad (13)$$

This is quite intuitive because the angular dispersion undergoes angular magnification at the later elements. We caution the reader that other elements of the combined system cannot be obtained from our analysis [Eq. (10)] unless they do not depend on the separations between the elements, as we have neglected the relevant propagation matrices between the elements.

3. Example: Modeling of Single-Prism-Grating Pulse Compressor

As an example, we can use Eqs. (11) and (12) to determine the magnification and angular dispersion introduced by a recently developed pulse compressor that involves quadruple passing a prism and grating; see Fig. 3.

A single-prism-grating compressor uses a combination of a prism and a reflection grating as the dispersive element. After the first pass through both of these elements, the beam is reflected precisely back using a corner cube, which inverts the beam and displaces it vertically for the second pass. After the first

two passes, a roof mirror returns the beam for its final two passes in the same fashion. In all, the device involves four passes through the grating and eight through the prism. Each pass through the prism-grating pair can be divided into three parts as the pulse goes through the prism (referred to as prism 1 in this section), the grating, and then the prism again (referred to as prism 2 in this section). The

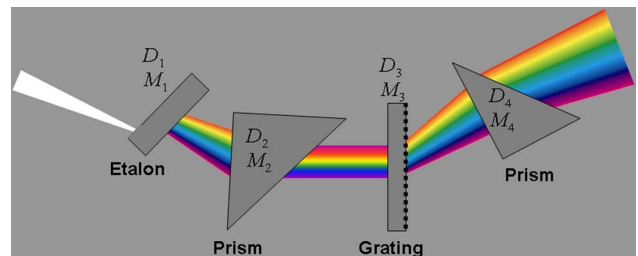


Fig. 2. (Color online) Sequence of dispersive optics with their respective dispersions and magnifications.

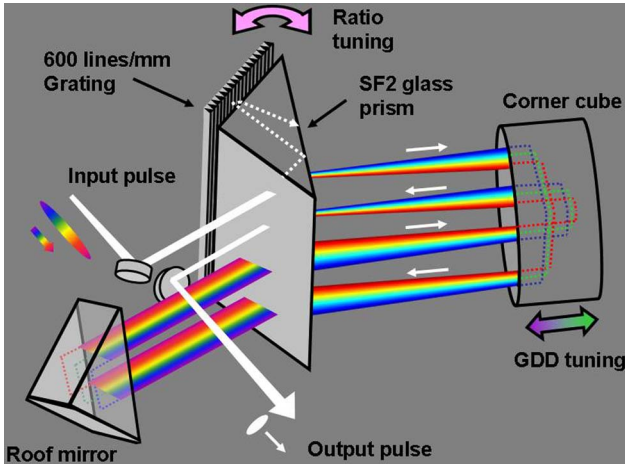


Fig. 3. (Color online) Schematic of a single-prism-grating pulse compressor.

magnification M on each pass is just the product of the three individual magnifications.

To perform this calculation, we note that inverting the beam or inverting the dispersive device induces sign reversals of certain elements in the corresponding K matrix, including those for the added spatial chirp, angular dispersion, pulse-front tilt and time versus angle:

$$\begin{bmatrix} M & B & 0 & E \\ 0 & 1/M & 0 & D \\ DM/\lambda_0 & H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} M & B & 0 & -E \\ 0 & 1/M & 0 & -D \\ -DM/\lambda_0 & -H & 1 & I \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (14)$$

From Eq. (11), the magnification of each prism-grating combination is given by

$$M = M_{\text{prism1}} M_{\text{grating}} M_{\text{prism2}}. \quad (15)$$

The angular dispersion D introduced on the first pass through the prism-grating pair is given by the dispersion equation, Eq. (12), which yields

$$D = \frac{D_{\text{prism1}}}{M_{\text{grating}} M_{\text{prism2}}} + \frac{D_{\text{grating}}}{M_{\text{prism2}}} + D_{\text{prism2}}. \quad (16)$$

Another useful result that we require and one that is easy to show is that, if a dispersive component of combination thereof has dispersion D and magnification M in the forward direction, then it has dispersion MD and magnification $1/M$ in the reverse direction. This can be verified by interchanging the input and the output angles in expressions for dispersion and magnification introduced by each device. Using this result and accounting for the beam inversion by the corner cube, the angular dispersion added on each pass through the prism-grating pair is

$$M_1 D_1 = -D_2 = -M_3 D_3 = D_4, \quad (17)$$

where D_i is the dispersion on the i th pass, and

$$M_1 = \frac{1}{M_2} = M_3 = \frac{1}{M_4}. \quad (18)$$

To calculate the total dispersion added by the compressor, we again use the same result as in Eq. (12), but this time for a sequence of prism-grating pairs as they are encountered by the pulse on each pass:

$$D_{\text{tot}} = \frac{D_1}{M_2 M_3 M_4} + \frac{D_2}{M_3 M_4} + \frac{D_3}{M_4} + D_4. \quad (19)$$

After substituting values from Eqs. (17) and (18), we find that the total angular dispersion added by the device is precisely zero for all angles of incidence, which is a requirement for a pulse compressor to operate properly. Additionally, the total magnification is unity, also appropriate for a pulse compressor's proper operation. To calculate the group-delay dispersion added by this compressor, we simply require the angular dispersion introduced in the first pass and the length of the path taken by the pulse before the second pass. Using Eq. (16), we can calculate the angular dispersion of the combination of elements, D , and then the GDD is given by

$$\text{GDD} = -\frac{2D^2 L}{\lambda_0}, \quad (20)$$

where the distance traveled by the pulse inside the dispersive element is assumed to be $\ll L$.

4. Conclusions

In conclusion, we have shown that the total angular dispersion due to multiple dispersive elements is the sum of the individual dispersions, with each divided by the total magnification occurring afterward (or multiplied by the total angular dispersion occurring afterward). This simple result yields a more intuitive understanding of the dispersive properties of several dispersive elements when used together and should greatly simplify the modeling of devices that contain such elements.

References

1. R. J. Niefert and J. B. Atkinson, "The design of achromatic prism beam expanders for pulsed dye lasers," *Opt. Commun.* **67**, 139–143 (1988).
2. D. C. Hanna, P. A. Kärkkäinen, and R. Wyatt, "A simple beam expander for frequency narrowing of dye lasers," *Opt. Quantum Electron.* **7**, 115–119 (1975).
3. I. Shoshan and U. P. Oppenheim, "The use of a diffraction grating as a beam expander in a dye laser cavity," *Opt. Commun.* **25**, 375–378 (1978).
4. R. Trebino, "Achromatic N-prism beam expanders: optimal configurations," *Appl. Opt.* **24**, 1130–1138 (1985).
5. F. J. Duarte, "Solid-state multiple-prism grating dye-laser oscillators," *Appl. Opt.* **33**, 3857–3860 (1994).

6. F. J. Duarte and J. A. Piper, "Prism preexpanded grazing-incidence grating cavity for pulsed dye lasers," *Appl. Opt.* **20**, 2113–2116 (1981).
7. R. L. Fork, C. H. B. Cruz, P. C. Becker, and C. V. Shank, "Compression of optical pulses to six femtoseconds by using cubic phase compensation," *Opt. Lett.* **12**, 483–485 (1987).
8. W. Dietel, J. J. Fontaine, and J. C. Diels, "Intracavity pulse compression with glass: a new method of generating pulses shorter than 60 fsec," *Opt. Lett.* **8**, 4–6 (1983).
9. R. L. Fork, O. E. Martinez, and J. P. Gordon, "Negative dispersion using pairs of prisms," *Opt. Lett.* **9**, 150–152 (1984).
10. O. Martinez, "3000 times grating compressor with positive group velocity dispersion: application to fiber compensation in 1.3–1.6 μm region," *IEEE J. Quantum Electron.* **23**, 59–64 (1987).
11. S. Akturk, X. Gu, M. Kimmel, and R. Trebino, "Extremely simple single-prism ultrashort-pulse compressor," *Opt. Express* **14**, 10101–10108 (2006).
12. S. Kane and J. Squier, "Grism-pair stretcher-compressor system for simultaneous second- and third-order dispersion compensation in chirped-pulse amplification," *J. Opt. Soc. Am. B* **14**, 661–665 (1997).
13. E. A. Gibson, D. M. Gaudiosi, H. C. Kapteyn, R. Jimenez, S. Kane, R. Huff, C. Durfee, and J. Squier, "Efficient reflection gratings for pulse compression and dispersion compensation of femtosecond pulses," *Opt. Lett.* **31**, 3363–3365 (2006).
14. V. Chauhan, P. Bowlan, J. Cohen, and R. Trebino, "Single-diffraction-grating and grism pulse compressors," *J. Opt. Soc. Am. B* **27**, 619–624 (2010).
15. F. J. Duarte and J. A. Piper, "Dispersion theory of multiple-prism beam expanders for pulsed dye lasers," *Opt. Commun.* **43**, 303–307 (1982).
16. F. A. Jenkins and H. E. White, *Fundamentals of Optics*, 3rd ed. (McGraw-Hill, 1957).
17. G. Chartier, *Introduction to Optics* (Springer, 2005).
18. M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University, 1999).
19. I. R. Kenyon, *Light Fantastic* (Oxford University, 2008), pp. 640–645.
20. J. K. Robertson, *Introduction to Physical Optics*, 3rd ed., University Physics Series (Van Nostrand, 1941).
21. G. S. Monk, *Light: Principles and Experiments*, 2nd ed. (Dover, 2000).
22. J. R. Meyer-Arendt, *Introduction to Classical and Modern Optics*, 4th ed. (Benjamin Cummings, 1994), pp. 480–482.
23. R. S. Longhurst, *Geometrical and Physical Optics*, 3rd ed. (Longman Group, 1974), pp. 677–685.
24. A. G. Kostenbauder, "Ray-pulse matrices: a rational treatment for dispersive optical systems," *IEEE J. Quantum Electron.* **26**, 1148–1157 (1990).